How Geometry implies that $\sum \frac{1}{k^2} = \frac{\pi^2}{6}$

Robert E. Greene

Write, for z∈ℂ, z not a non-positive real number, $L(z) = \ln|z| + i\theta$ where $z = re^{i\theta}$, $\theta \in (-\pi, \pi)$. By a theorem of Euclid, $\operatorname{Im} L(1 + e^{i\theta}) = \frac{\theta}{2}$. (See figure 1)

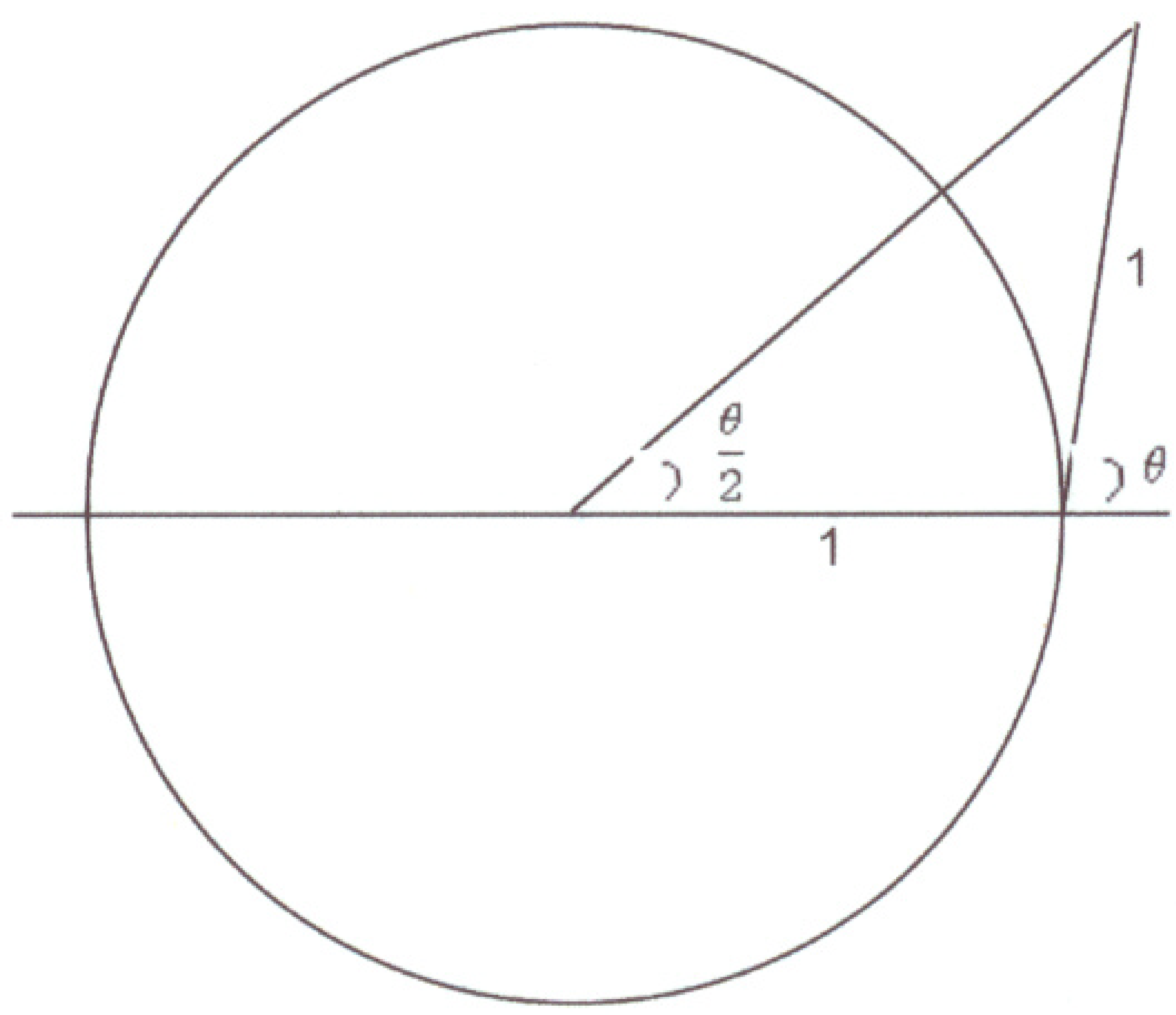
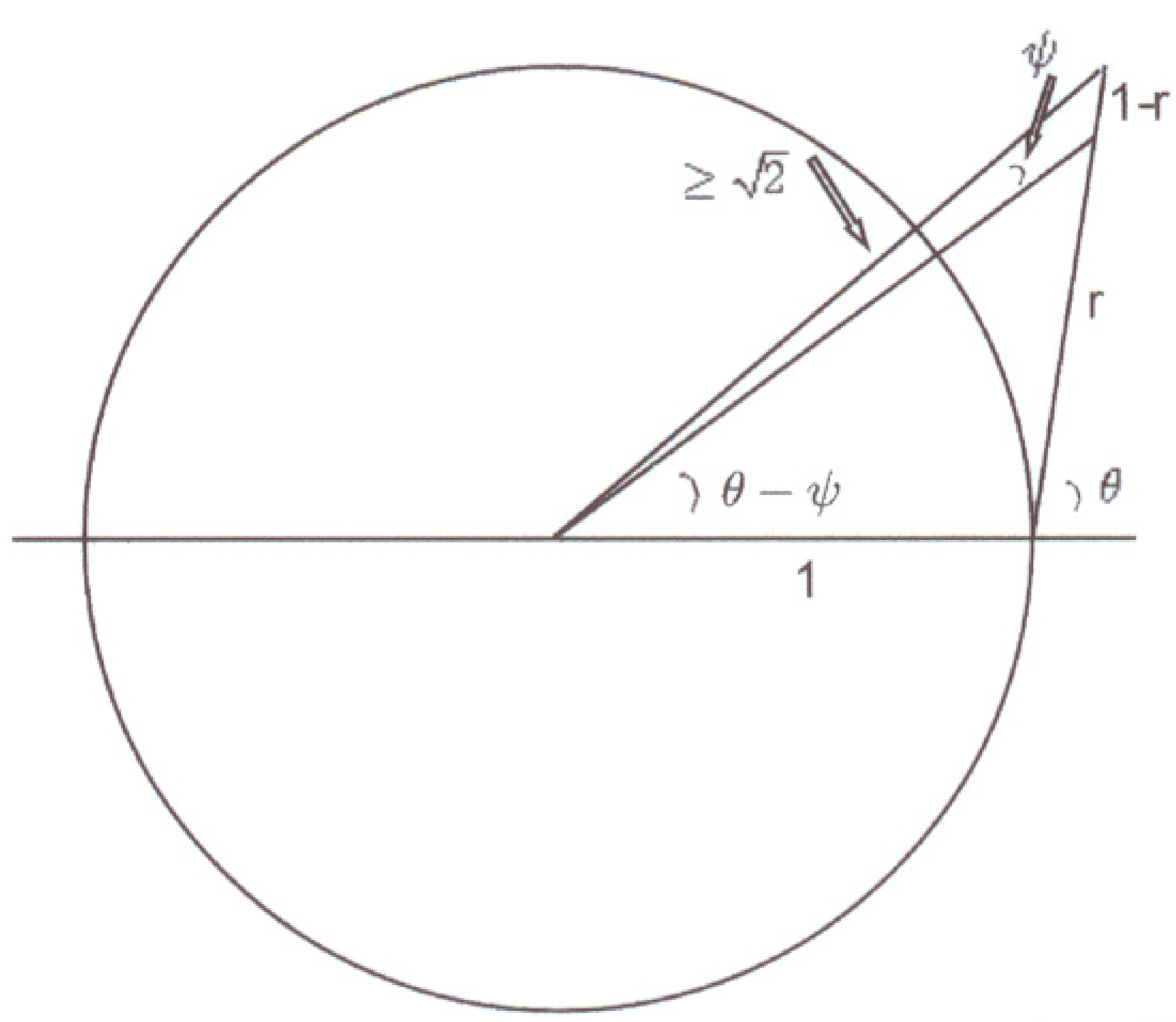


Figure 1

By further examination of the geometry, $\lim L(1 + re^{i\theta}) = \frac{\theta}{2}$ as $r \to 1^-$ and the limit is uniform with respect to $\theta \in \left[0, \frac{\pi}{2}\right]$. (See figure 2)



$\psi \to 0^+$ as $r \to 1^-$ uniformly for $\theta \in \left[0, \frac{\pi}{2}\right]$

Figure 2

Note that if $r \in [0,1)$ and $\theta \in (-\pi, \pi)$ then $L(1+re^{is}) = \int_0^r \frac{e^{i\theta}}{1+te^{i\theta}}\,dt$ by the Fundamental Theorem of Calculus applied to real and imaginary parts (or by complex analysis). Thus if $\theta \in \left[0, \frac{\pi}{2}\right]$, with lim indicating the limit as $r \to 1^-$ and by using the geometric series expansion of $\frac{e^{is}}{1+te^{is}}$:

$$\begin{aligned}
\int_0^\theta \frac{s}{2}dt = \frac{\theta^2}{4} &= \lim \int_0^\theta \operatorname{Im}\left[L(1+re^{is})\right]ds \\
&= \lim \int_0^\theta \left(\int_0^r \frac{e^{is}}{1+te^{is}}\,dt \right) ds \\
&= \lim \int_0^\theta \left(\int_0^r \sum_{k=1}^{\infty} \operatorname{Im}\left((-1)^{k-1} e^{iks} t^{k-1} \right) dt \right) ds \\
&= \lim \int_0^\theta \sum_{k=1}^{\infty} (-1)^{k-1} k^{-1} \sin(ks) r^k \, ds \\
&= \lim \left[\left(\sum_{k=1}^{\infty} (-1)^{k-1} r^k k^{-2} \right) - \left(\sum_{k=1}^{\infty} (-1)^{k-1} r^k k^{-2} \cos k\theta \right) \right] \\
&= \sum_{k=1}^{\infty} (-1)^{k-1} k^{-2} - \sum_{k=1}^{\infty} (-1)^{k-1} k^{-2} \cos k\theta
\end{aligned}$$

Put $\theta = \frac{\pi}{2}$. This gives $\frac{\pi^2}{16} = \sum_k (-1)^{k-1} k^{-2} - 2^{-2} \sum_k (-1)^{k-1} k^{-2}$ $= \frac{3}{4} \sum_k (-1)^{k-1} k^{-2}$. Hence $\sum_k (-1)^{k-1} k^{-2} = \frac{\pi^2}{12}$

That $\sum \frac{1}{k^2} = \frac{\pi^2}{6}$ follows by the familiar algebraic observation that $\sum_k k^{-2} - (2 \cdot 2^{-2}) \sum_k k^{-2} = \sum_k (-1)^{k-1} k^{-2}$.

All interchanges of sums, limits and integrals are justified by standard results on uniform convergence and comparison tests. (M tests)

Robert E. Greene
Department of Mathematics
University of California, Los Angeles
Los Angeles, California 90095-1555
greene@math.ucla.edu